Growing a stochastic magnetic field in a turbulent fluid plasma: the Fokker–Planck approach
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Literature

- Rincon (2019), *Dynamo theories*. A review of small-scale and large-scale dynamos. Section 3.4 develops the Kazantsev model, with the spectral Fokker–Planck equation and its solutions in § 3.4.8.
- Schekochihin et al. (2002), *Spectra and Growth Rates of Fluctuating Magnetic Fields in the Kinematic Dynamo Theory with Large Magnetic Prandtl Numbers*. Section 2 derives the spectral equation and Green's-function solutions. Appendix A gives the isotropic Fourier transforms and solenoidal correlation tensors used to connect real-space and spectral descriptions.

1 Magnetic dynamos

Magnetic dynamos are ubiquitous across many scales in the Universe. They describe the transfer of energy from fluid motions into the magnetic field. In this lecture, I will move through small-scale dynamo theory where one seeks closures to transfer terms of the third order correlator $\langle uBB \rangle$. These lecture notes take inspiration from Rincon (2019) and Brandenburg and Ntormousi (2023) and Alex Schekochihin's notes on kinetic and fluid plasmas. I should apologize in advance that the citation list is not intended to be exhaustive.

1.1 Goals for this lecture

- Derive the Kazantsev model for the spectrum of magnetic field fluctuations in the presence of random velocity gradients.

2 Small-scale dynamo (kinematic only)

2.1 Magnetic energy spectrum evolution

In component form, the induction equation is

$$\partial_t B_i = \partial_j(u_i B_j - u_j B_i) + \eta \nabla^2 B_i. \tag{1}$$

Introducing the Fourier transform,

$$\hat{B}_i(\boldsymbol{k}, t) = \int \mathrm{d}^3x \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} B_i(\boldsymbol{x}, t), \tag{2}$$

the induction equation becomes

$$\partial_t \hat{B}_i(\boldsymbol{k}) = ik_j \int \frac{\mathrm{d}^3k' \, \mathrm{d}^3k''}{(2\pi)^3} \delta(\boldsymbol{k} - \boldsymbol{k}' - \boldsymbol{k}'') \Big[\hat{u}_i(\boldsymbol{k}')\hat{B}_j(\boldsymbol{k}'') - \hat{u}_j(\boldsymbol{k}')\hat{B}_i(\boldsymbol{k}'')\Big] - \eta k^2 \hat{B}_i(\boldsymbol{k}). \tag{3}$$

The magnetic energy associated with mode $\boldsymbol{k}$ is

$$E_B(\boldsymbol{k}, t) \equiv \frac{1}{2} |\hat{\boldsymbol{B}}(\boldsymbol{k}, t)|^2. \tag{4}$$

Its evolution is therefore

$$\partial_t E_B(\boldsymbol{k}, t) = \mathrm{Re}\left[\hat{B}_i^\dagger(\boldsymbol{k}) \partial_t \hat{B}_i(\boldsymbol{k})\right] \tag{5}$$

$$= \mathrm{Re}\left\{ik_j \hat{B}_i^\dagger(\boldsymbol{k}) \int \frac{\mathrm{d}^3k' \, \mathrm{d}^3k''}{(2\pi)^3} \delta(\boldsymbol{k} - \boldsymbol{k}' - \boldsymbol{k}'') \Big[\hat{u}_i(\boldsymbol{k}')\hat{B}_j(\boldsymbol{k}'') - \hat{u}_j(\boldsymbol{k}')\hat{B}_i(\boldsymbol{k}'')\Big]\right\} - 2\eta k^2 E_B(\boldsymbol{k}, t). \tag{6}$$

To pass from the mode-by-mode budget to an isotropic shell budget, define the one-dimensional magnetic energy spectrum by integrating over the spherical shell $|\boldsymbol{k}| = k$:

$$M(k, t) = \frac{1}{2} \int k^2 \, \mathrm{d}\Omega_k \, |\hat{\boldsymbol{B}}(\boldsymbol{k}, t)|^2, \tag{7}$$

so that

$$M(k, t) = \int k^2 \, \mathrm{d}\Omega_k \, E_B(\boldsymbol{k}, t). \tag{8}$$

Differentiating at fixed k gives

$$\partial_t M(k, t) = \int k^2 \, \mathrm{d}\Omega_k \, \partial_t E_B(\boldsymbol{k}, t), \tag{9}$$

and substituting the mode-energy equation yields the exact shell-integrated budget

$$\partial_t M(k, t) = \mathcal{T}_B(k, t) - 2\eta k^2 M(k, t). \tag{10}$$

It is useful to isolate the triadic integrand

$$T_B(\boldsymbol{k}; \boldsymbol{k}', \boldsymbol{k}'', t) \equiv \mathrm{Re}\left\{ik_j \hat{B}_i^\dagger(\boldsymbol{k}) \delta(\boldsymbol{k} - \boldsymbol{k}' - \boldsymbol{k}'') \Big[\hat{u}_i(\boldsymbol{k}')\hat{B}_j(\boldsymbol{k}'') - \hat{u}_j(\boldsymbol{k}')\hat{B}_i(\boldsymbol{k}'')\Big]\right\}, \tag{11}$$

where

$$\mathcal{T}_B(k, t) = \int k^2 \, \mathrm{d}\Omega_k \int \frac{\mathrm{d}^3k' \, \mathrm{d}^3k''}{(2\pi)^3} T_B(\boldsymbol{k}; \boldsymbol{k}', \boldsymbol{k}'', t). \tag{12}$$

The delta function enforces the triad condition $\boldsymbol{k} = \boldsymbol{k}' + \boldsymbol{k}''$, so $\mathcal{T}_B(k, t)$ is the total induction contribution into the shell of radius k after summing over all velocity–magnetic–magnetic triads that end on that shell. The resistive term is local in k, while the inductive term is nonlocal in Fourier space and depends on mixed third-order correlations. In the kinematic problem this term describes both redistribution among magnetic wavenumbers and net amplification by the prescribed velocity field, so it is not closed as written. A statistical closure is therefore required to obtain a tractable evolution equation for $M(k, t)$.

2.2 The Kazantsev (1968) $k^{3/2}$ model

2.2.1 Closure of the third-order correlators

The exact spectral evolution equation

$$\partial_t M(k,t) = \mathcal{T}_B(k,t) - 2\eta k^2 M(k,t) \tag{13}$$

is not closed, since the inductive transfer term $\mathcal{T}_B$ depends on mixed third-order moments of the form $\langle uBB\rangle$. Thus, the evolution of the second-order magnetic correlator depends on higher-order correlations, and a closure is required.

The Kazantsev–Kraichnan model provides such a closure by replacing the velocity field with a prescribed random process that is Gaussian, homogeneous, isotropic, non-helical, and δ-correlated in time. Its two-point covariance is

$$\langle u_i(\boldsymbol{x},t)u_j(\boldsymbol{x}',t')\rangle = \kappa_{ij}(\boldsymbol{r})\,\delta(t-t'), \qquad \boldsymbol{r} \equiv \boldsymbol{x}' - \boldsymbol{x}, \tag{14}$$

with the isotropic tensor decomposition

$$\kappa_{ij}(\boldsymbol{r}) = \kappa_N(r)\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right) + \kappa_L(r)\frac{r_i r_j}{r^2}. \tag{15}$$

For an incompressible flow, $\partial_i u_i = 0$, so the transverse and longitudinal correlators are not independent:

$$\kappa_N(r) = \kappa_L(r) + \frac{r}{2}\frac{d\kappa_L(r)}{dr}. \tag{16}$$

In the kinematic regime, the velocity is prescribed and receives no magnetic back reaction. The magnetic field nevertheless depends on the velocity history through the induction equation, so the two fields are not generally statistically independent. The δ-correlation in time implies that the velocity has no memory, so the magnetic field experiences a Markovian sequence of random stretching events.

The most natural object to evolve is the equal-time magnetic two-point correlator,

$$H_{ij}(\boldsymbol{r},t) \equiv \langle B_i(\boldsymbol{x},t)B_j(\boldsymbol{x}',t)\rangle\,. \tag{17}$$

Isotropy implies the analogous decomposition

$$H_{ij}(\boldsymbol{r},t) = H_N(r,t)\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right) + H_L(r,t)\frac{r_i r_j}{r^2}, \tag{18}$$

and $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ further gives

$$H_N(r,t) = H_L(r,t) + \frac{r}{2}\frac{\partial H_L(r,t)}{\partial r}. \tag{19}$$

Thus the whole isotropic problem reduces to the single scalar function $H_L(r,t)$.

2.2.2 Gaussian integration by parts for one random variable

A scalar example isolates the probabilistic identity that underlies the closure. A zero-mean Gaussian random variable z with variance σ^2 has probability density

$$p(z) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\!\left(-\frac{z^2}{2\sigma^2}\right). \tag{20}$$

Differentiating this density gives

$$\frac{dp(z)}{dz} = -\frac{z}{\sigma^2}p(z), \qquad zp(z) = -\sigma^2\frac{dp(z)}{dz}. \tag{21}$$

The function $F(z)$ can be any sufficiently regular function for which the expectations below exist and the boundary term vanishes,

$$\lim_{z\to\pm\infty} F(z)p(z) = 0. \tag{22}$$

In particular, F need not be linear or Gaussian. Substituting the density derivative into the expectation and integrating by parts gives

$$\begin{aligned}\langle zF(z)\rangle &= \int_{-\infty}^{\infty} dz\, zF(z)p(z)\\ &= -\sigma^2 \int_{-\infty}^{\infty} dz\, F(z)\frac{dp(z)}{dz}\\ &= -\sigma^2\Big[F(z)p(z)\Big]_{-\infty}^{\infty} + \sigma^2 \int_{-\infty}^{\infty} dz\, p(z)\frac{dF(z)}{dz}\\ &= \sigma^2 \left\langle \frac{dF(z)}{dz}\right\rangle . \end{aligned} \tag{23}$$

The integration by parts moves the derivative from the Gaussian probability density onto F. This is an exact identity under the stated conditions.

In the dynamo problem, the role of z is played by the Gaussian velocity field, while the role of $F(z)$ is played by the magnetic product $B_i(\boldsymbol{x},t)B_j(\boldsymbol{x}',t)$. This product is a functional of the velocity history, so the field version of Equation 23 uses a functional derivative.

2.2.3 From Gaussian integration by parts to the Kazantsev equation

The magnetic fields are evaluated at $\boldsymbol{x}$ and $\boldsymbol{x}'$, with $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}$. Gaussian integration by parts requires an integral over the velocity history. We denote its dummy position and time by $\boldsymbol{x}''$ and t''. They are integration variables, not additional magnetic-field positions:

$$\langle u_m(\boldsymbol{x},t)F[\boldsymbol{u}]\rangle = \int dt'' \int d^3x''\, \kappa_{mn}(\boldsymbol{x}''-\boldsymbol{x})\delta(t-t'') \left\langle \frac{\delta F[\boldsymbol{u}]}{\delta u_n(\boldsymbol{x}'',t'')}\right\rangle , \tag{24}$$

where $F[\boldsymbol{u}] = B_i(\boldsymbol{x},t)B_j(\boldsymbol{x}',t)$. We evaluate the magnetic response using the induction equation. A velocity perturbation $\epsilon\boldsymbol{v}$ acting for Δt changes the field by

$$\partial_\epsilon B_i^{(\epsilon)}(\boldsymbol{x},t+\Delta t)\Big|_{\epsilon=0} = \Delta t\, \partial_{x_a}(v_iB_a - v_aB_i) + O(\Delta t^2). \tag{25}$$

Choosing $v_i(\boldsymbol{x}) = \delta_{in}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'')$ localizes the perturbation. The corresponding equal-time functional response is

$$\frac{\delta B_i(\boldsymbol{x},t)}{\delta u_n(\boldsymbol{x}'',t'')}\bigg|_{t''=t} = \frac{1}{2}\partial_{x_a}\left[\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'')(\delta_{in}B_a - \delta_{an}B_i)\right]. \tag{26}$$

The factor $1/2$ includes the causal endpoint weight for symmetric temporal regularization. With that weight already included, the temporal delta function in Equation 24 selects this response without another factor $1/2$. Suppressing the common time t, the product rule is

$$\frac{\delta[B_i(\boldsymbol{x})B_j(\boldsymbol{x}')]}{\delta u_n(\boldsymbol{x}'',t'')} = B_j(\boldsymbol{x}')\frac{\delta B_i(\boldsymbol{x})}{\delta u_n(\boldsymbol{x}'',t'')} + B_i(\boldsymbol{x})\frac{\delta B_j(\boldsymbol{x}')}{\delta u_n(\boldsymbol{x}'',t'')}. \tag{27}$$

Substituting the product rule into Equation 24 first gives

$$\begin{aligned}&\langle u_m(\boldsymbol{x},t)B_i(\boldsymbol{x},t)B_j(\boldsymbol{x}',t)\rangle\\ &\quad= \int dt'' \int d^3x''\, \kappa_{mn}(\boldsymbol{x}''-\boldsymbol{x})\delta(t-t'')\\ &\qquad\times\left\langle B_j(\boldsymbol{x}',t)\frac{\delta B_i(\boldsymbol{x},t)}{\delta u_n(\boldsymbol{x}'',t'')} + B_i(\boldsymbol{x},t)\frac{\delta B_j(\boldsymbol{x}',t)}{\delta u_n(\boldsymbol{x}'',t'')}\right\rangle . \end{aligned} \tag{28}$$

We now insert Equation 26 at each magnetic position. The temporal delta function selects the equal-time responses, and the magnetic averages become H_{aj}, H_{ij}, and H_{ia}. Suppressing t, the integral is explicitly

$$\begin{aligned}&\langle u_m(\boldsymbol{x})B_i(\boldsymbol{x})B_j(\boldsymbol{x}')\rangle\\ &\quad= \frac{1}{2}\int d^3x''\, \kappa_{mn}(\boldsymbol{x}''-\boldsymbol{x})\Big\{\partial_{x_a}\left[\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'')(\delta_{in}H_{aj} - \delta_{an}H_{ij})\right]\\ &\qquad\qquad + \partial_{x'_a}\left[\delta^{(3)}(\boldsymbol{x}'-\boldsymbol{x}'')(\delta_{jn}H_{ia} - \delta_{an}H_{ij})\right]\Big\}. \end{aligned} \tag{29}$$

The two derivatives act only on their square brackets. The first bracket is the response at $\boldsymbol{x}$, and the second is the response at $\boldsymbol{x}'$. The undifferentiated delta functions perform the integrals as

$$\int d^3x''\,\kappa_{mn}(\boldsymbol{x}''-\boldsymbol{x})\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'') = \kappa_{mn}(\boldsymbol{0}),$$
$$\int d^3x''\,\kappa_{mn}(\boldsymbol{x}''-\boldsymbol{x})\delta^{(3)}(\boldsymbol{x}'-\boldsymbol{x}'') = \kappa_{mn}(\boldsymbol{r}). \tag{30}$$

For a derivative of the second delta function, integration by parts gives

$$\begin{aligned}
&\int d^3x''\,\kappa_{mn}(\boldsymbol{x}''-\boldsymbol{x})\partial_{x'_a}\delta^{(3)}(\boldsymbol{x}'-\boldsymbol{x}'') \\
&\quad= -\int d^3x''\,\kappa_{mn}(\boldsymbol{x}''-\boldsymbol{x})\partial_{x''_a}\delta^{(3)}(\boldsymbol{x}'-\boldsymbol{x}'') \\
&\quad= \int d^3x''\,[\partial_{x''_a}\kappa_{mn}(\boldsymbol{x}''-\boldsymbol{x})]\delta^{(3)}(\boldsymbol{x}'-\boldsymbol{x}'') = \partial_{r_a}\kappa_{mn}(\boldsymbol{r}).
\end{aligned} \tag{31}$$

The same operation at $\boldsymbol{x}$ gives the covariance gradient at $\boldsymbol{r}=\boldsymbol{0}$, which vanishes by evenness. Performing these integrals in Equation 29 and contracting $\kappa_{mn}\delta_{in}=\kappa_{mi}$, $\kappa_{mn}\delta_{jn}=\kappa_{mj}$, and $\kappa_{mn}\delta_{an}=\kappa_{ma}$ gives

$$\begin{aligned}
&\langle u_m(\boldsymbol{x})B_i(\boldsymbol{x})B_j(\boldsymbol{x}')\rangle \\
&\quad= \frac{1}{2}\left[\kappa_{mi}(\boldsymbol{0})\partial_{x_a}H_{aj} - \kappa_{ma}(\boldsymbol{0})\partial_{x_a}H_{ij}\right] \\
&\qquad+ \frac{1}{2}\partial_{x'_a}\left[\kappa_{mj}(\boldsymbol{r})H_{ia} - \kappa_{ma}(\boldsymbol{r})H_{ij}\right].
\end{aligned} \tag{32}$$

Homogeneity gives $\partial_{x_a}=-\partial_{r_a}$ and $\partial_{x'_a}=\partial_{r_a}$. Solenoidality gives $\partial_{r_a}H_{aj}=\partial_{r_a}H_{ia}=0$ and $\partial_{r_a}\kappa_{ma}=0$. Thus the same result can be written entirely in the established separation coordinate:

$$\langle u_m(\boldsymbol{x})B_i(\boldsymbol{x})B_j(\boldsymbol{x}')\rangle = \frac{1}{2}\left[(\kappa_{ma}(\boldsymbol{0})-\kappa_{ma}(\boldsymbol{r}))\partial_{r_a}H_{ij} + (\partial_{r_a}\kappa_{mj}(\boldsymbol{r}))H_{ia}\right]. \tag{33}$$

The mixed moment with velocity at $\boldsymbol{x}'$ follows by exchanging $(\boldsymbol{x},i)$ and $(\boldsymbol{x}',j)$, which sends $\boldsymbol{r}\to-\boldsymbol{r}$. Differentiating the magnetic product in time gives

$$\begin{aligned}
\partial_t H_{ij} &= \partial_{x_a}\langle(u_iB_a - u_aB_i)(\boldsymbol{x})B_j(\boldsymbol{x}')\rangle \\
&\quad+ \partial_{x'_a}\langle B_i(\boldsymbol{x})(u_jB_a - u_aB_j)(\boldsymbol{x}')\rangle + 2\eta\nabla_r^2 H_{ij}.
\end{aligned} \tag{34}$$

Using Equation 32 and homogeneity, $\partial_{x_a}=-\partial_{r_a}$ and $\partial_{x'_a}=\partial_{r_a}$, combines the coincident and separated velocity covariances into

$$\begin{aligned}
D_{ij}(\boldsymbol{r}) &= 2\eta\delta_{ij} + \kappa_{ij}(\boldsymbol{0}) - \kappa_{ij}(\boldsymbol{r}), \\
\partial_t H_{ij} &= \partial_{r_a}\partial_{r_b}\left[D_{ab}H_{ij} + D_{ij}H_{ab} - D_{ib}H_{aj} - D_{aj}H_{ib}\right].
\end{aligned} \tag{35}$$

Finally, we substitute the isotropic tensors, use $H_N = H_L + (r/2)\partial_r H_L$ and $D_N = D_L + (r/2)\partial_r D_L$, and project along $n_i = r_i/r$. The angular derivatives are $\partial_{r_a}n_i = (\delta_{ai}-n_an_i)/r$. This gives (Rincon, 2019, § 3.4.4)

$$\begin{aligned}
\partial_t H_L &= D_L\partial_r^2 H_L + \left(\frac{4D_L}{r} + \partial_r D_L\right)\partial_r H_L \\
&\quad+ \left(\partial_r^2 D_L + \frac{4}{r}\partial_r D_L\right)H_L, \qquad D_L(r) = 2\eta + \kappa_L(0) - \kappa_L(r).
\end{aligned} \tag{36}$$

2.2.4 The spectral Fokker–Planck equation

At $\mathrm{Pm}\gg 1$, the resistive scale lies below the viscous scale, so the velocity is smooth across the magnetic structures. Its covariance has the small-separation expansion

$$\begin{aligned}
\kappa_L(r) &= \kappa_0 - \frac{\kappa_2}{4}r^2 + O(r^4), \\
\kappa_N(r) &= \kappa_0 - \frac{\kappa_2}{2}r^2 + O(r^4).
\end{aligned} \tag{37}$$

Here $\kappa_0 = \kappa_L(0)$ and $\kappa_2 = -2\,\partial_r^2\kappa_L(r)\big|_{r=0}$. Thus κ_2 measures the curvature of the longitudinal velocity covariance with respect to separation. The first derivative vanishes at $r = 0$ by evenness.

Physically, κ_2 measures the strength of the velocity gradients integrated over their correlation time. Differentiating the covariance at both positions gives $\langle \partial_{x_a} u_i(\boldsymbol{x}, t)\partial_{x'_b} u_j(\boldsymbol{x}', t')\rangle = -\partial_{r_a}\partial_{r_b}\kappa_{ij}(\boldsymbol{r})\delta(t-t')$. Contracting $a = b$ and $i = j$ at $\boldsymbol{r} = \boldsymbol{0}$ yields

$$\langle \boldsymbol{\nabla u}(\boldsymbol{x}, t) : \boldsymbol{\nabla u}(\boldsymbol{x}, t')\rangle = \frac{15}{2}\kappa_2\delta(t-t'),$$
$$\kappa_2 = \frac{2}{15}\int_{-\infty}^{\infty} d\tau\ \langle \boldsymbol{\nabla u}(\boldsymbol{x}, t) : \boldsymbol{\nabla u}(\boldsymbol{x}, t+\tau)\rangle\,. \tag{38}$$

For a flow with finite correlation time, the corresponding estimate is $\kappa_2 \sim \tau_{\mathrm{corr}}\langle \boldsymbol{\nabla u} : \boldsymbol{\nabla u}\rangle$. Thus κ_2 has units of inverse time, rather than the inverse-time-squared units of $\boldsymbol{\nabla u} : \boldsymbol{\nabla u}$. The diffusivity is then $D_L = 2\eta + \kappa_2 r^2/4$. Substitution into Equation 36 gives

$$\partial_t H_L = 2\eta\left(\partial_r^2 H_L + \frac{4}{r}\partial_r H_L\right) + \frac{\kappa_2}{4}\left(r^2\partial_r^2 H_L + 6r\partial_r H_L + 10H_L\right). \tag{39}$$

Angular integration of the isotropic solenoidal magnetic covariance relates H_L to M through

$$H_L(r, t) \propto \int_0^\infty M(k, t)\mathcal{K}(kr)\,dk, \qquad \mathcal{K}(s) = \frac{\sin s - s\cos s}{s^3}. \tag{40}$$

The proportionality constant depends only on the Fourier normalization and cancels from the evolution equation. The kernel satisfies

$$\left(\partial_r^2 + \frac{4}{r}\partial_r\right)\mathcal{K}(kr) = -k^2\mathcal{K}(kr), \qquad r\partial_r\mathcal{K}(kr) = k\partial_k\mathcal{K}(kr). \tag{41}$$

Consequently, the resistive term transforms to $-2\eta k^2 M$. For the stretching term, integration by parts transfers both k derivatives from the kernel onto M:

$$\begin{aligned}\int_0^\infty & M\left(k^2\partial_k^2\mathcal{K} + 6k\partial_k\mathcal{K} + 10\mathcal{K}\right)dk \\ &= \int_0^\infty \mathcal{K}\left[\partial_k^2(k^2M) - 6\partial_k(kM) + 10M\right]dk \\ &= \int_0^\infty \mathcal{K}\left(k^2\partial_k^2 M - 2k\partial_k M + 6M\right)dk,\end{aligned} \tag{42}$$

where the boundary terms vanish. With $\gamma = 5\kappa_2/4$, comparison of the transforms gives the spectral Fokker–Planck equation (Rincon, 2019, § 3.4.8),

$$\boxed{\partial_t M = \frac{\gamma}{5}\left(k^2\partial_k^2 M - 2k\partial_k M + 6M\right) - 2\eta k^2 M.} \tag{43}$$

The parameter $\gamma \sim \delta u_\nu/\ell_\nu$ measures the viscous-scale strain rate. It is distinct from the magnetic-energy growth rate γ_E. This spectral equation applies to magnetic modes with $k \gg k_\nu$, where k_ν is the viscous wavenumber. Random stretching amplifies the spectrum and spreads it in $\ln k$, while resistivity damps modes at large k.

2.2.5 Diffusion-free evolution and the Kazantsev spectrum

Before the spectrum reaches the resistive scale, $2\eta k^2 M$ is negligible. We use the logarithmic coordinate $y = \ln(k/k_\nu)$, for which

$$k\partial_k = \partial_y, \qquad k^2\partial_k^2 = \partial_y^2 - \partial_y. \tag{44}$$

The diffusion-free equation is therefore

$$\partial_t M = \frac{\gamma}{5}\left(\partial_y^2 M - 3\partial_y M + 6M\right). \tag{45}$$

Writing $M = e^{3y/2}\Psi(y,t)$ removes the first derivative and gives

$$\partial_t \Psi = \frac{\gamma}{5}\partial_y^2 \Psi + \frac{3\gamma}{4}\Psi. \tag{46}$$

This is diffusion in logarithmic wavenumber with uniform amplification. The initial spectrum is $M_0(k) = M(k,0)$. The Green's function evolves each initial shell at k_0, and integration over y_0 sums their contributions. Since $dy_0 = d\ln k_0 = dk_0/k_0$, the general solution is

$$M(k,t) = e^{3\gamma t/4}\sqrt{\frac{5}{4\pi\gamma t}}\int_0^\infty \frac{dk_0}{k_0} M_0(k_0)\left(\frac{k}{k_0}\right)^{3/2} \exp\left[-\frac{5\ln^2(k/k_0)}{4\gamma t}\right]. \tag{47}$$

For a simple example, we place the initial magnetic energy $E_{\rm mag,0}$ in a narrow shell at k_s:

$$M_0(k_0) = E_{\rm mag,0}\delta(k_0 - k_s), \qquad \int_0^\infty M_0(k_0)\,dk_0 = E_{\rm mag,0}. \tag{48}$$

We choose $k_\nu \ll k_s \ll k_\eta$: the seed wavelength is long compared with the resistive scale, while remaining in the smooth-flow regime. The delta function performs the integral in Equation 47, giving

$$\boxed{M(k,t) = \frac{E_{\rm mag,0}}{k_s} e^{3\gamma t/4}\sqrt{\frac{5}{4\pi\gamma t}}\left(\frac{k}{k_s}\right)^{3/2} \exp\left[-\frac{5\ln^2(k/k_s)}{4\gamma t}\right].} \tag{49}$$

The factor $1/k_s$ comes from the logarithmic integration measure: $\int (dk_0/k_0)\delta(k_0 - k_s) f(k_0) = f(k_s)/k_s$.

The $k^{3/2}$ factor is the Kazantsev spectrum. The full solution also has a time-dependent envelope, so its local slope approaches $3/2$ where that envelope varies slowly. The logarithmic diffusion kernel has variance $2\gamma t/5$. We characterize its evolution by the spectral rms wavenumber, the peak of $M(k,t)$, and the total magnetic energy. The Gaussian integrals in Equation 49 give $\int k^2 M\,dk = E_{\rm mag,0}k_s^2 e^{24\gamma t/5}$ and $\int M\,dk = E_{\rm mag,0}e^{2\gamma t}$, so

$$\boxed{\begin{aligned} k_{\rm rms}(t) &\equiv \left[\frac{\int_0^\infty k^2 M(k,t)\,dk}{\int_0^\infty M(k,t)\,dk}\right]^{1/2} = k_s e^{7\gamma t/5}, \\ k_{\rm peak}(t) &\equiv \arg\max_k M(k,t) = k_s e^{3\gamma t/5}, \\ E_{\rm mag}(t) &\equiv \int_0^\infty M(k,t)\,dk = E_{\rm mag,0}e^{2\gamma t}. \end{aligned}} \tag{50}$$

All three grow exponentially, with γ setting the rate. The rms wavenumber grows faster than the peak because its k^2 weighting emphasizes the expanding high-k tail. The magnetic-energy growth rate is $\gamma_E = 2\gamma$. These results hold during diffusion-free evolution, before resistivity truncates the spectrum.

2.2.6 The resistively truncated spectrum

Resistivity arrests the drift near

$$k_\eta = \sqrt{\frac{\gamma}{10\eta}} \sim {\rm Pm}^{1/2} k_\nu. \tag{51}$$

We seek a fixed spectral shape with an exponentially growing amplitude,

$$M(k,t) = e^{\lambda\gamma t}\left(\frac{k}{k_\eta}\right)^{3/2} f(q), \qquad q = \frac{k}{k_\eta}. \tag{52}$$

Substitution into Equation 43 gives

$$q^2\frac{d^2 f}{dq^2} + q\frac{df}{dq} - \left(q^2 + 5\lambda - \frac{15}{4}\right) f = 0. \tag{53}$$

The solution that decays at large q is the modified Bessel function $K_\mu(q)$, with $\mu^2 = 5\lambda - 15/4$. Matching at the low-k end selects λ. In the large-scale-separation limit, $\lambda \to 3/4$ and $\mu \to 0$, giving

$$\boxed{M(k,t) \propto e^{3\gamma t/4} \left(\frac{k}{k_\eta}\right)^{3/2} K_0\left(\frac{k}{k_\eta}\right).} \tag{54}$$

Thus the spectrum has a $k^{3/2}$ factor with a slowly varying logarithmic correction below the cutoff, since $K_0(q) \sim -\ln(q/2) - \gamma_{\rm Euler}$ for $q \ll 1$. Above it, $K_0(q) \sim \sqrt{\pi/(2q)}e^{-q}$ produces an exponential falloff. The spectral shape is now fixed, so each mode and the total magnetic energy grow at the same rate, $\gamma_E = \lambda\gamma \simeq 3\gamma/4$.

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