Magnetic-energy evolution and spectrum under shear winding

Notes for *The kinematic Kelvin–Helmholtz instability-driven dynamo: implications for merging compact objects*

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These notes derive the magnetic-energy evolution and ideal winding spectrum for stage 1 of the Kelvin–Helmholtz instability (KHI)-driven dynamo. The mean shear winds the initial cross-layer field into a streamwise field, whose energy grows as t^2 in the ideal limit. Winding also transfers magnetic energy to higher k. Resistivity removes magnetic energy, while viscosity broadens the shear layer and weakens further winding. We evaluate the model coefficients from the initial magnetic field and shear profiles. Ideal winding is exact for the prescribed stationary shear. The non-ideal winding model extends the initial diffusion corrections through an approximate description of layer broadening.

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1 Shear geometry and definitions

The shear layer lies in the yz plane. The streamwise velocity points along y and varies across x:

$$\boldsymbol{u}(x,t) = U(x,t)\boldsymbol{e}_y. \tag{1}$$

x is the cross-layer coordinate, y points along the flow, and z is perpendicular to the initial shear-flow plane, xy. The magnetic field may vary in all three directions.

For velocity jump ΔU and initial half-width δ_0, the shear time, Reynolds numbers, and magnetic Prandtl number are

$$t_\Delta = \frac{\delta_0}{\Delta U}, \qquad \mathrm{Re}_\Delta = \frac{\Delta U \delta_0}{\nu} = \frac{\delta_0^2}{\nu t_\Delta}, \qquad \mathrm{Rm}_\Delta = \frac{\Delta U \delta_0}{\eta} = \frac{\delta_0^2}{\eta t_\Delta}, \qquad \mathrm{Pm} = \frac{\nu}{\eta} = \frac{\mathrm{Rm}_\Delta}{\mathrm{Re}_\Delta}. \tag{2}$$

ν is the constant kinematic viscosity and η is the constant Ohmic magnetic diffusivity. Both have units of length squared per time, and $\mathrm{Rm}_\Delta = \mathrm{Re}_\Delta \mathrm{Pm}$.

For the periodic volume V, we define

$$\langle f \rangle = \frac{1}{V}\int_V f \, dV, \qquad E_{\mathrm{mag}}(t) = \frac{1}{2}\left\langle B_x^2 + B_y^2 + B_z^2 \right\rangle, \qquad E_{\mathrm{mag},0} = \frac{1}{2}\left\langle B_0^2 \right\rangle. \tag{3}$$

We set the magnetic permeability to unity. A subscript 0 denotes an initial quantity. Here E_{mag} is the volume-averaged magnetic-energy density. The manuscript uses volume-integrated energies, VE_{mag}, giving the same normalized energies and model coefficients at fixed V. The spectrum $E_{\mathrm{mag}}(k,t)$ is energy per unit wavenumber:

$$E_{\mathrm{mag}}(t) = \int_0^\infty E_{\mathrm{mag}}(k,t)\, dk. \tag{4}$$

Repeated Cartesian indices imply summation over x, y, z (Einstein notation). Sums over distinct magnetic modes remain explicit.

k is angular wavenumber, with wavelength $2\pi/k$. For mode number $n = kL/(2\pi)$, $E_{\mathrm{mag}}(n,t)\, dn = E_{\mathrm{mag}}(k,t)\, dk$, so $E_{\mathrm{mag}}(n,t) = (2\pi/L)E_{\mathrm{mag}}(k,t)$. This change of units preserves the spectral slope.

1.1 Model assumptions

To isolate winding, we retain only the prescribed planar shear and neglect the imposed transverse perturbation, billow roll-up, three-dimensional motions, and Lorentz-force backreaction. The field therefore evolves without changing the velocity.

Since $\boldsymbol{\nabla} \cdot \boldsymbol{u} = \partial_y U = 0$, the mass equation reduces to

$$\partial_t \rho + U \partial_y \rho = 0. \tag{5}$$

For initial density $\rho_0(x)$, this gives $\rho(x,t) = \rho_0(x)$. The prescribed flow is incompressible, although the simulations are compressible. Volume averages use periodic boundaries. Isolated-layer integrals extend over $-\infty < x < \infty$, with vanishing velocity derivatives at infinity.

1.2 Initial profiles

The initial velocity and density profiles determine the winding and broadening coefficients. For two layers centered at $x = \pm L/4$,

$$H(x) = \frac{1}{2}\left[\tanh\left(\frac{x + L/4}{\delta_0}\right) - \tanh\left(\frac{x - L/4}{\delta_0}\right)\right], \tag{6}$$

$$U_0(x) = \Delta U\left[H(x) - \frac{1}{2}\right], \qquad \rho_0(x) = \rho_{\rm out}\left[1 + \frac{5}{3}H(x)\right]. \tag{7}$$

$\rho_{\rm in}/\rho_{\rm out} = 8/3$ and $\delta_0/L = 0.025$, giving a separation of $20\delta_0$ and negligible overlap of the shear profiles. The initial magnetic field is three-dimensional and divergence-free, $\boldsymbol{\nabla}\cdot\boldsymbol{B}_0 = 0$. Its structure determines the normalized winding coefficients, while its overall amplitude cancels.

2 Ideal and non-ideal magnetic energy evolution

We first derive ideal winding, then include resistive losses and viscous layer broadening. The resulting coefficients determine the non-ideal winding model.

2.1 Ideal winding

Ideal induction gives the generated field and its magnetic energy before diffusion is included.

2.1.1 Component induction equations

With constant η and $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, induction is

$$\partial_t\boldsymbol{B} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{B} = (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \boldsymbol{B}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) + \eta\nabla^2\boldsymbol{B}. \tag{8}$$

For Equation 1, $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{B} = U\partial_y\boldsymbol{B}$, $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$, and $(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{u} = B_x(\partial_x U)\boldsymbol{e}_y$. Hence

$$(\partial_t + U\partial_y)B_x = \eta\nabla^2 B_x, \tag{9}$$

$$(\partial_t + U\partial_y)B_y = (\partial_x U)B_x + \eta\nabla^2 B_y, \tag{10}$$

$$(\partial_t + U\partial_y)B_z = \eta\nabla^2 B_z. \tag{11}$$

The shear generates B_y from B_x, while B_x and B_z are transported and diffused.

We first set $\eta = 0$ and hold the shear fixed at $U_0(x)$. A fluid element starting at (X, Y, Z) follows

$$x = X, \qquad y = Y + U_0(X)t, \qquad z = Z. \tag{12}$$

Along this trajectory, $\partial_t + U_0\partial_y = d/dt$, and Equation 9 give

$$\frac{dB_x}{dt} = 0, \qquad \frac{dB_y}{dt} = (\partial_x U_0)(X)B_{x0}(X, Y, Z), \qquad \frac{dB_z}{dt} = 0. \tag{13}$$

The shear and B_{x0} are constant along this trajectory. Integration gives

$$B_x(x, y, z, t) = B_{x0}(x, y - U_0(x)t, z), \tag{14}$$

$$B_y(x, y, z, t) = B_{y0}(x, y - U_0(x)t, z) + t(\partial_x U_0)(x)B_{x0}(x, y - U_0(x)t, z), \tag{15}$$

$$B_z(x, y, z, t) = B_{z0}(x, y - U_0(x)t, z). \tag{16}$$

These solutions are exact for the stationary shear and any divergence-free initial field.

2.1.2 Magnetic-energy polynomial

The component solution determines the total energy through its squared magnitude. At a given starting position,

$$B^2(t) = B_{x0}^2 + [B_{y0} + t(\partial_x U_0)B_{x0}]^2 + B_{z0}^2 \tag{17}$$

$$= B_0^2 + 2t(\partial_x U_0)B_{x0}B_{y0} + t^2(\partial_x U_0)^2 B_{x0}^2. \tag{18}$$

The translation in y preserves volume and its periodic average. Averaging the squared field and dividing by $\langle B_0^2 \rangle$ gives

$$\boxed{\frac{E_{\mathrm{mag}}(t)}{E_{\mathrm{mag},0}} = 1 + a\frac{t}{t_\Delta} + b\left(\frac{t}{t_\Delta}\right)^2,} \tag{19}$$

where

$$a = 2t_\Delta \frac{\langle (\partial_x U_0)B_{x0}B_{y0} \rangle}{\langle B_0^2 \rangle}, \tag{20}$$

$$b = t_\Delta^2 \frac{\langle (\partial_x U_0)^2 B_{x0}^2 \rangle}{\langle B_0^2 \rangle}. \tag{21}$$

The linear term is the cross term between the initial and generated streamwise fields and can have either sign. The quadratic term is the generated field's energy and is nonnegative. Both arise from ideal winding.

2.1.3 Seed dependence of a and b

The coefficients a, b retain correlations between the seed and the layer. For a statistically isotropic, homogeneous seed independent of the layer, the ensemble means satisfy $\langle B_{x0}B_{y0} \rangle = 0$ and $\langle B_{x0}^2 \rangle = \langle B_0^2 \rangle /3$. They give

$$a = 0, \qquad b = \frac{t_\Delta^2}{3} \left\langle (\partial_x U_0)^2 \right\rangle. \tag{22}$$

A finite realization of the seed can have nonzero component correlations and uneven B_{x0}^2 across the layer. Evaluating Equation 20 for the initial field retains these features.

2.2 Resistive losses

Winding creates magnetic gradients that increase resistive losses. Following a single magnetic mode determines how these gradients grow and how diffusion accumulates over time.

2.2.1 Magnetic modes in a locally linear shear

A region with approximately constant velocity gradient can be represented by

$$U(x) = U_c + (\partial_x U)x. \tag{23}$$

Following the uniform translation removes U_c. The remaining $\partial_x U$ is constant in space and time.

A magnetic mode is a spatial Fourier component:

$$\boldsymbol{B}(\boldsymbol{x}, t) = \widehat{\boldsymbol{B}}(t) \exp(i[k_x(t)x + k_y(t)y + k_z(t)z]). \tag{24}$$

A real field pairs this magnetic mode with its complex conjugate. Applying $\partial_t + U\partial_y$ to the advected phase gives

$$[\dot{k}_x + (\partial_x U)k_y]x + \dot{k}_y y + \dot{k}_z z = 0. \tag{25}$$

Each coordinate has an independent coefficient, giving

$$\frac{dk_x}{dt} = -(\partial_x U)k_y, \qquad \frac{dk_y}{dt} = 0, \qquad \frac{dk_z}{dt} = 0. \tag{26}$$

For initial wavevector $\boldsymbol{p} = (p_x, p_y, p_z)$,

$$k_x = p_x - t(\partial_x U)p_y, \qquad k_y = p_y, \qquad k_z = p_z. \tag{27}$$

Shear changes the cross-layer wavelength while leaving the y, z wavelengths fixed. At late times, $|k_x|$ grows as t.

Applying $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ to this magnetic mode multiplies it by $-k^2$. Its amplitudes therefore obey

$$\frac{d\widehat{B}_x}{dt} = -\eta k^2 \widehat{B}_x, \tag{28}$$

$$\frac{d\widehat{B}_y}{dt} = (\partial_x U)\widehat{B}_x - \eta k^2 \widehat{B}_y, \tag{29}$$

$$\frac{d\widehat{B}_z}{dt} = -\eta k^2 \widehat{B}_z. \tag{30}$$

Multiplication by $\exp[\eta \int_0^t k^2(t')dt']$ cancels the common decay term, leaving the ideal amplitude equations. Their solution is

$$\widehat{\boldsymbol{B}}(t) = \begin{pmatrix} \widehat{B}_{x0} \\ \widehat{B}_{y0} + t(\partial_x U)\widehat{B}_{x0} \\ \widehat{B}_{z0} \end{pmatrix} \exp\left[-\eta \int_0^t k^2(t')dt'\right]. \tag{31}$$

Squaring the amplitudes gives the energy damping $\exp[-2\eta \int_0^t k^2(t')dt']$.

2.2.2 Accumulated diffusion

The decay depends on the accumulated squared wavenumber. Squaring Equation 27 gives

$$k^2(t') = p_x^2 + p_y^2 + p_z^2 - 2t'(\partial_x U)p_x p_y + t'^2(\partial_x U)^2 p_y^2. \tag{32}$$

Termwise integration gives

$$\int_0^t k^2(t')dt' = \overbrace{p^2 t}^{\text{initial gradients}} - \underbrace{(\partial_x U)p_x p_y t^2}_{\text{initial mode orientation}} + \overbrace{\frac{1}{3}(\partial_x U)^2 p_y^2 t^3}^{\text{gradients created by winding}}. \tag{33}$$

The t term describes diffusion at the initial wavelength. The t^2 term depends on whether the initial tilt first lengthens or shortens that wavelength. The t^3 term describes the gradients generated by winding. Their sum is nonnegative because it integrates k^2.

Equation 27–(33) are exact for a stationary linear shear. Applying them locally to the tanh layer approximates its varying gradient.

2.2.3 Initial-field diffusion coefficient α

The initial resistive energy loss fixes α. Multiplying the induction equations by their corresponding field components and adding gives

$$\partial_t \frac{B^2}{2} + U\partial_y \frac{B^2}{2} = (\partial_x U) B_x B_y + \eta B_i \partial_j \partial_j B_i. \tag{34}$$

The y derivative integrates to zero over a periodic box, because U has no y dependence. For the resistive contribution, the product rule gives

$$B_i \partial_j \partial_j B_i = \partial_j (B_i \partial_j B_i) - (\partial_j B_i)(\partial_j B_i). \tag{35}$$

The total derivative integrates to zero. The remaining loss can be expressed through the current, $\boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{B}$, because

$$J_i J_i = (\partial_j B_i)(\partial_j B_i) - (\partial_j B_i)(\partial_i B_j). \tag{36}$$

The second term vanishes after volume averaging:

$$\langle (\partial_j B_i)(\partial_i B_j) \rangle = \langle \partial_j (B_i \partial_i B_j) \rangle - \langle B_i \partial_i (\partial_j B_j) \rangle = 0, \tag{37}$$

$$\left\langle |\boldsymbol{\nabla} \boldsymbol{B}|^2 \right\rangle = \left\langle J^2 \right\rangle. \tag{38}$$

The boundary contribution vanishes by periodicity, and $\partial_j B_j = 0$. Thus $|\boldsymbol{\nabla} \boldsymbol{B}|^2$ and J^2 have equal volume averages, although they can differ pointwise. The energy equation is

$$\boxed{\frac{dE_{\mathrm{mag}}}{dt} = \langle (\partial_x U) B_x B_y \rangle - \eta \left\langle J^2 \right\rangle.} \tag{39}$$

The second term is the magnetic-energy loss due to resistivity. At $t = 0$, its fractional contribution is

$$\frac{1}{E_{\mathrm{mag},0}} \frac{dE_{\mathrm{mag}}}{dt}\bigg|_{\text{resistive, }0} = -2\eta \frac{\langle J_0^2 \rangle}{\langle B_0^2 \rangle}. \tag{40}$$

We define

$$\alpha = 2\delta_0^2 \frac{\langle J_0^2 \rangle}{\langle B_0^2 \rangle}. \tag{41}$$

The factor $\exp(-\alpha \eta t / \delta_0^2)$ matches this initial fractional loss rate. It approximates subsequent diffusion of the initial field by assigning one decay rate to its different wavelengths.

2.2.4 Winding-created gradient coefficient β

β describes the additional losses from winding-created gradients. The cubic term in Equation 33 multiplies a magnetic mode's energy by

$$\exp\left[-\frac{2}{3} \eta (\partial_x U)^2 p_y^2 t^3 \right]. \tag{42}$$

The generated energy weights each magnetic mode by $(\partial_x U)^2 |\widehat{B}_{x0}|^2$. Expanding its cubic damping factor supplies two further shear powers and p_y^2, represented by $|\partial_y B_{x0}|^2$. The resulting relative correction is

$$1 - \frac{2\eta t^3}{3} \frac{\langle (\partial_x U_0)^4 |\partial_y B_{x0}|^2 \rangle}{\langle (\partial_x U_0)^2 B_{x0}^2 \rangle} + \cdots. \tag{43}$$

Matching this correction to $1 - \beta\eta t^3/(\delta_0^2 t_\Delta^2) + \cdots$ gives

$$\beta = \frac{2\delta_0^2 t_\Delta^2}{3} \frac{\langle (\partial_x U_0)^4 |\partial_y B_{x0}|^2 \rangle}{\langle (\partial_x U_0)^2 B_{x0}^2 \rangle}. \tag{44}$$

The corresponding loss factor is $\exp[-\beta\eta t^3/(\delta_0^2 t_\Delta^2)]$.

β is nonnegative and dimensionless. Its numerator weights the winding-created gradients by the generated magnetic energy. One exponential approximates the distribution of local loss rates, and the orientation-dependent t^2 term in Equation 33 is omitted. The energy model applies this additional cubic loss only to the quadratic winding term. In the exact magnetic-mode solution, all three energy contributions share the same damping factor.

2.3 Resistive spreading coefficient κ_η

Resistivity also acts on the layer profile itself. Even for uniform B_{x0}, the generated B_y varies across x because $\partial_x U_0$ varies. Spreading this profile lowers its energy and determines κ_η.

For fixed U_0 and locally constant B_{x0}, the generated profile varies only across x and obeys

$$\partial_t B_{y,\mathrm{wound}} = B_{x0}\partial_x U_0 + \eta \partial_x^2 B_{y,\mathrm{wound}}, \qquad B_{y,\mathrm{wound}}(x,0) = 0. \tag{45}$$

This isolates resistive spreading of the generated field across the layer from the winding-created gradients described by Equation 27.

At $t = 0$, $\partial_t B_{y,\mathrm{wound}} = B_{x0}\partial_x U_0$ and $\partial_t^2 B_{y,\mathrm{wound}} = \eta B_{x0}\partial_x^3 U_0$. The time expansion is

$$B_{y,\mathrm{wound}} = B_{x0}\left[t\partial_x U_0 + \frac{\eta t^2}{2}\partial_x^3 U_0 + \cdots\right]. \tag{46}$$

Squaring, retaining the first correction, and integrating across the layer gives

$$\int B_{y,\mathrm{wound}}^2 dx = B_{x0}^2 t^2 \left[\int (\partial_x U_0)^2 dx + \eta t \int (\partial_x U_0)(\partial_x^3 U_0) dx + \cdots\right]. \tag{47}$$

Integration by parts moves one derivative from the third derivative:

$$\int (\partial_x U_0)(\partial_x^3 U_0) dx = \Big[(\partial_x U_0)(\partial_x^2 U_0)\Big]_{-\infty}^{\infty} - \int (\partial_x^2 U_0)^2 dx = -\int (\partial_x^2 U_0)^2 dx. \tag{48}$$

The derivatives vanish far from the layer. Dividing by the leading generated energy gives

$$\frac{\int B_{y,\mathrm{wound}}^2 dx}{B_{x0}^2 t^2 \int (\partial_x U_0)^2 dx} = 1 - \kappa_\eta \frac{\eta t}{\delta_0^2} + \cdots, \qquad \kappa_\eta = \delta_0^2 \frac{\int (\partial_x^2 U_0)^2 dx}{\int (\partial_x U_0)^2 dx}. \tag{49}$$

2.3.1 Layer integrals for a tanh profile

Evaluating the gradient integrals fixes κ_η for the prescribed tanh profile. For an isolated layer centered at x_c,

$$U_0 = \frac{\Delta U}{2} \tanh\left(\frac{x - x_c}{\delta_0}\right). \tag{50}$$

Reversing the layer leaves the squared derivatives unchanged. The dimensionless integration variable $z = \tanh[(x - x_c)/\delta_0]$ gives

$$dx = \frac{\delta_0\, dz}{1 - z^2}, \qquad \partial_x U_0 = \frac{\Delta U}{2\delta_0}(1 - z^2), \qquad \partial_x^2 U_0 = -\frac{\Delta U}{\delta_0^2} z(1 - z^2). \tag{51}$$

Consequently,

$$\int (\partial_x U_0)^2 dx = \frac{(\Delta U)^2}{4\delta_0} \int_{-1}^{1} (1 - z^2) dz = \frac{(\Delta U)^2}{3\delta_0}, \tag{52}$$

$$\int (\partial_x^2 U_0)^2 dx = \frac{(\Delta U)^2}{\delta_0^3} \int_{-1}^{1} z^2 (1 - z^2) dz = \frac{4(\Delta U)^2}{15\delta_0^3}. \tag{53}$$

Their ratio gives

$$\boxed{\kappa_\eta = \frac{4}{5}.} \tag{54}$$

Two separated layers double both integrals, leaving their ratio unchanged. This calculation neglects layer overlap, finite-box tails, and variation of B_{x0}^2 across each layer.

2.4 Viscous broadening coefficient κ_ν

Viscosity changes the generating shear itself. Its effect on magnetic energy therefore depends on the shear accumulated over time.

2.4.1 Evolution of the prescribed shear

For constant kinematic viscosity, the relevant viscous stress is $\rho_0 \nu \partial_x U$. The y momentum equation for this flow reduces to

$$\rho_0 \partial_t U = \partial_x(\rho_0 \nu \partial_x U), \qquad \partial_t U = \nu \rho_0^{-1} \partial_x(\rho_0 \partial_x U). \tag{55}$$

Both the y pressure gradient and $U\partial_y U$ vanish. Density remains $\rho_0(x)$. We neglect Lorentz-force backreaction and cross-layer motions driven by viscous heating. With constant kinematic viscosity, the density gradients remain in Equation 55.

2.4.2 Time-integrated shear

The generated field retains the history of the changing shear. For $\eta = 0$ and evolving U, a fluid element remains at fixed x and moves by $\int_0^t U(x, t')dt'$ along y. Integrating induction gives

$$B_{y,\mathrm{wound}} = B_{x0} \int_0^t \partial_x U(x, t')dt'. \tag{56}$$

B_{x0} is evaluated at the element's initial position. The time integral includes the earlier, stronger shear that $t\partial_x U(x, t)$ would omit.

Expanding Equation 55 about the initial profile gives

$$U(x, t') = U_0(x) + \nu t' \rho_0^{-1} \partial_x(\rho_0 \partial_x U_0) + \cdots, \tag{57}$$

$$\int_0^t \partial_x U(x, t')dt' = t\partial_x U_0 + \frac{\nu t^2}{2} \partial_x \left[\rho_0^{-1} \partial_x(\rho_0 \partial_x U_0)\right] + \cdots. \tag{58}$$

Again treating B_{x0} as approximately constant across the layer, squaring and integrating gives a relative generated-energy correction

$$1 + \nu t \frac{\int (\partial_x U_0)\partial_x[\rho_0^{-1}\partial_x(\rho_0 \partial_x U_0)]dx}{\int (\partial_x U_0)^2 dx} + \cdots . \tag{59}$$

Squaring the field cancels the time-integral factor $1/2$ against the cross-term factor of two.

2.4.3 Integration by parts of the shear profile

Integration by parts separates shear spreading from the density correction. The derivative in Equation 59 expands to

$$\partial_x[\rho_0^{-1}\partial_x(\rho_0 \partial_x U_0)] = \partial_x^3 U_0 + (\partial_x \ln \rho_0)\partial_x^2 U_0 + (\partial_x^2 \ln \rho_0)\partial_x U_0. \tag{60}$$

The first contribution integrates to $-\int (\partial_x^2 U_0)^2 dx$. The identity $(\partial_x U_0)(\partial_x^2 U_0) = \frac{1}{2}\partial_x[(\partial_x U_0)^2]$ gives the second. All boundary terms vanish because the shear and density derivatives decay:

$$\int (\partial_x \ln \rho_0)(\partial_x U_0)(\partial_x^2 U_0)dx \tag{61}$$

$$= \frac{1}{2}\int (\partial_x \ln \rho_0)\partial_x[(\partial_x U_0)^2]dx = -\frac{1}{2}\int (\partial_x^2 \ln \rho_0)(\partial_x U_0)^2 dx. \tag{62}$$

The third contribution adds one copy of the density integral, leaving

$$\int (\partial_x U_0)\partial_x[\rho_0^{-1}\partial_x(\rho_0 \partial_x U_0)]dx \tag{63}$$

$$= -\int (\partial_x^2 U_0)^2 dx + \frac{1}{2}\int (\partial_x^2 \ln \rho_0)(\partial_x U_0)^2 dx. \tag{64}$$

Thus Equation 59 is $1 - \kappa_\nu \nu t/\delta_0^2 + \cdots$, with

$$\boxed{\kappa_\nu = \kappa_\eta - \frac{\delta_0^2}{2}\frac{\int (\partial_x^2 \ln \rho_0)(\partial_x U_0)^2 dx}{\int (\partial_x U_0)^2 dx}.} \tag{65}$$

Uniform density removes the second term and gives $\kappa_\nu = \kappa_\eta = 4/5$.

2.4.4 Density correction for the prescribed layer

The remaining density integral fixes κ_ν for the contrast $8/3$. For the rising layer, the same z substitution gives

$$\rho_0 = \rho_{\text{out}}\frac{11 + 5z}{6}, \qquad \partial_x \ln \rho_0 = \frac{1}{\delta_0}\frac{5(1 - z^2)}{11 + 5z}. \tag{66}$$

The falling layer gives the same integral after reversing z. Since $\partial_x = (1 - z^2)\partial_z/\delta_0$, Equation 65 becomes

$$\kappa_\nu = \frac{4}{5} - \frac{3}{8}\int_{-1}^{1}(1 - z^2)^2 \frac{d}{dz}\left[\frac{5(1 - z^2)}{11 + 5z}\right] dz. \tag{67}$$

The derivative appearing here is

$$\frac{d}{dz}\left[\frac{5(1-z^2)}{11+5z}\right] = -\frac{25z^2+110z+25}{(11+5z)^2}. \tag{68}$$

Substitution of $v = 11+5z$ changes the limits to $6, 16$ and the integrand to

$$-\frac{(v^2-22v+96)^2(v^2-96)}{3{,}125v^2}\,dv. \tag{69}$$

Expanding and integrating term by term gives the antiderivative

$$-\frac{1}{3{,}125}\left[\frac{v^5}{5} - 11v^4 + \frac{580v^3}{3} - 55{,}680v + 405{,}504\ln v + \frac{884{,}736}{v}\right]. \tag{70}$$

Evaluating it at 16 and 6, and substituting into Equation 67, gives

$$\boxed{\kappa_\nu = \frac{152{,}064}{3{,}125}\ln\frac{8}{3} - \frac{29{,}298}{625} = 0.8508222496\ldots.} \tag{71}$$

Only the prescribed density and velocity profiles enter κ_ν.

2.5 Seed-dependent coefficients a, b, α, β

We evaluate a, b, α, β from the prescribed magnetic modes, retaining their correlations with the layer. The layer integrals independently determine κ_η, κ_ν. The required inputs are

Coefficient	Initial quantities required	Defining expression
a	$\boldsymbol{B}_0$ and $\partial_x U_0$	Equation 20
b	$\boldsymbol{B}_0$ and $(\partial_x U_0)^2$	Equation 20
α	$\boldsymbol{B}_0$ and $\boldsymbol{J}_0$	Equation 41
β	B_{x0}, $\partial_y B_{x0}$ and $\partial_x U_0$	Equation 44
κ_η	U_0 only	Equation 49
κ_ν	U_0 and ρ_0	Equation 65

2.5.1 Representation by magnetic modes and volume averages

A mode representation reduces the required volume averages to finite sums. The initial field is

$$B_{i0}(\boldsymbol{x}) = \sum_{\boldsymbol{n}} c_i(\boldsymbol{n}) e^{2\pi i \boldsymbol{n}\cdot\boldsymbol{x}/L}, \qquad c_i(-\boldsymbol{n}) = c_i(\boldsymbol{n})^*. \tag{72}$$

$\boldsymbol{n}$ is an integer wavevector, $c_i(\boldsymbol{n})$ its prescribed component amplitude, and the asterisk denotes complex conjugation. Paired magnetic modes make the field real.

Averaging over the periodic volume removes products of magnetic modes with different wavevectors. The initial mean squared field and current are therefore

$$\left\langle B_0^2 \right\rangle = \sum_{\boldsymbol{n}} c_i(\boldsymbol{n}) c_i(\boldsymbol{n})^*, \tag{73}$$

$$\left\langle J_0^2 \right\rangle = \sum_{\boldsymbol{n}} \left|\frac{2\pi\boldsymbol{n}}{L}\right|^2 c_i(\boldsymbol{n}) c_i(\boldsymbol{n})^*. \tag{74}$$

For $\boldsymbol{p} = 2\pi\boldsymbol{n}/L$, the current amplitude is $i\boldsymbol{p} \times \boldsymbol{c}$. Since $\boldsymbol{p} \cdot \boldsymbol{c} = 0$, its squared magnitude is $|\boldsymbol{p} \times \boldsymbol{c}|^2 = p^2|\boldsymbol{c}|^2$, giving the second average.

A shear-weighted average also retains pairs of magnetic modes with different x wavenumbers. For a real profile $W(x)$, we calculate

$$\widehat{W}(\Delta p_x) = \frac{1}{L}\int_{-L/2}^{L/2} W(x)e^{i\Delta p_x x}dx. \tag{75}$$

Multiplying two initial fields and integrating their phases over y, z gives

$$\langle W(x)B_{i0}B_{j0}\rangle = \sum_{\substack{\boldsymbol{n},\boldsymbol{m} \\ n_y=m_y,\ n_z=m_z}} c_i(\boldsymbol{n})c_j(\boldsymbol{m})^*\widehat{W}(\Delta p_x), \qquad \Delta p_x = \frac{2\pi(n_x - m_x)}{L}. \tag{76}$$

The conditions below the sum come from the y, z averages. The remaining x integral is precisely $\widehat{W}(\Delta p_x)$. The same conditions and the same Δp_x apply to each sum over paired magnetic modes below.

2.5.2 Seed correlations for a and b

For a, we substitute $W = \partial_x U_0$ and the fixed components x, y into Equation 76. For b, we substitute $W = (\partial_x U_0)^2$ and the fixed components x, x. Their definitions in Equation 20 become

$$a = \frac{2t_\Delta}{\langle B_0^2\rangle} \sum_{\substack{\boldsymbol{n},\boldsymbol{m} \\ n_y=m_y,\ n_z=m_z}} c_x(\boldsymbol{n})c_y(\boldsymbol{m})^*\widehat{(\partial_x U_0)}(\Delta p_x), \tag{77}$$

$$b = \frac{t_\Delta^2}{\langle B_0^2\rangle} \sum_{\substack{\boldsymbol{n},\boldsymbol{m} \\ n_y=m_y,\ n_z=m_z}} c_x(\boldsymbol{n})c_x(\boldsymbol{m})^*\widehat{(\partial_x U_0)^2}(\Delta p_x). \tag{78}$$

For the prescribed seed, these give

$$a = -0.00106582141222, \qquad b = 0.00345363295126. \tag{79}$$

a measures the shear-weighted correlation of B_{x0}, B_{y0}, while b measures the shear-squared-weighted B_{x0} energy.

2.5.3 Initial current and α

The ratio $\langle J_0^2\rangle / \langle B_0^2\rangle$ fixes α without a shear weight. Substitution of Equation 73 into Equation 41 gives

$$\alpha = 2\delta_0^2 \frac{\sum_{\boldsymbol{n}} |2\pi\boldsymbol{n}/L|^2\, c_i(\boldsymbol{n})c_i(\boldsymbol{n})^*}{\sum_{\boldsymbol{n}} c_i(\boldsymbol{n})c_i(\boldsymbol{n})^*} = 0.236542265983. \tag{80}$$

2.5.4 Along-flow gradients and β

For β, we require $|\partial_y B_{x0}|^2$. The derivative of each paired magnetic mode supplies one factor of $2\pi n_y/L$, giving $(2\pi n_y/L)^2$ because $n_y = m_y$. Substitution of the fourth and second powers of the

shear into Equation 44 gives

$$\beta = \frac{2\delta_0^2 t_\Delta^2}{3} \frac{\sum_{\substack{\boldsymbol{n},\boldsymbol{m} \\ n_y=m_y,\ n_z=m_z}} (2\pi n_y/L)^2 c_x(\boldsymbol{n}) c_x(\boldsymbol{m})^* (\widehat{\partial_x U_0})^4 (\Delta p_x)}{\sum_{\substack{\boldsymbol{n},\boldsymbol{m} \\ n_y=m_y,\ n_z=m_z}} c_x(\boldsymbol{n}) c_x(\boldsymbol{m})^* (\widehat{\partial_x U_0})^2 (\Delta p_x)}. \tag{81}$$

The prescribed seed gives $\beta = 0.00629780985658$.

2.5.5 Shear-profile integrals

Evaluation of a, b, β requires the first, second, and fourth powers of the shear profile. The squared-shear integral also enters the denominator of β. The coefficient α requires no profile integral. For the tanh layer, these integrals can be evaluated analytically. With $\operatorname{sech}\xi = 1/\cosh\xi$, the dimensionless argument ζ gives

$$I_m(\zeta) = \int_{-\infty}^{\infty} \operatorname{sech}^m \xi \, e^{i\zeta\xi} d\xi, \tag{82}$$

$$I_2(\zeta) = \frac{\pi\zeta}{\sinh(\pi\zeta/2)}, \qquad I_2(0) = 2. \tag{83}$$

The higher even powers follow from

$$I_{m+2}(\zeta) = \frac{\zeta^2 + m^2}{m(m+1)} I_m(\zeta), \qquad m = 2, 4, 6. \tag{84}$$

Differentiating $\operatorname{sech}^m \xi$ twice gives $m^2 \operatorname{sech}^m \xi - m(m+1)\operatorname{sech}^{m+2}\xi$. Integrating that derivative against $e^{i\zeta\xi}$ gives $-\zeta^2 I_m$. Equating these expressions gives the stated relation.

For a layer centered at x_c and even m,

$$(\widehat{\partial_x U_0})^m (\Delta p_x) \simeq \frac{\delta_0}{L} \left(\frac{\Delta U}{2\delta_0}\right)^m e^{i\Delta p_x x_c} I_{2m}(\delta_0 \Delta p_x). \tag{85}$$

The cases $m = 2, 4$ require I_4, I_8. The first power, $\partial_x U_0$, instead uses I_2 and the sign of the layer. We add the two layers with opposite signs for $\partial_x U_0$ and the same sign for its even powers, neglecting their small overlap as before. Together with $\kappa_\eta = 4/5$ and $\kappa_\nu = 0.850822249592$, these coefficients determine the non-ideal winding model. All remain fixed when comparing simulations.

2.6 Non-ideal winding model

The calculated coefficients specify the initial corrections. Combining them into a finite-time model also requires an approximation for the evolving layer width.

2.6.1 Combined initial broadening

For $\nu t/\delta_0^2 \ll 1$ and $\eta t/\delta_0^2 \ll 1$, the relative correction to the quadratic energy is

$$1 - \frac{t}{\delta_0^2}(\kappa_\nu \nu + \kappa_\eta \eta) + \cdots. \tag{86}$$

The viscous and resistive corrections add at first order. The coefficients α and β separately describe diffusion of the initial field and losses from winding-created gradients. The initial correction does not uniquely determine later evolution. Extending it requires an assumption about how the layer broadens.

2.6.2 Finite-time broadening approximation

To extend the initial correction to finite time, we approximate the generated field in one layer by a profile of width $w(t)$ and fixed shape:

$$B_{y,\mathrm{wound}}(x,t) \simeq B_{x0} t \frac{\Delta U}{w(t)} f\left(\frac{x - x_c}{w(t)}\right). \tag{87}$$

The assumed shape f has unit integral and remains fixed as the width changes. Thus $\int B_{y,\mathrm{wound}} dx = B_{x0} t \Delta U$, while

$$\int B_{y,\mathrm{wound}}^2 dx \simeq B_{x0}^2 t^2 \frac{(\Delta U)^2}{w(t)} \int f^2(\xi) d\xi. \tag{88}$$

Using $\xi = (x - x_c)/w(t)$ gives $dx = w(t) d\xi$ in this integral. The constant $\int f^2 d\xi$ cancels when dividing by the ideal result with width δ_0, leaving the factor $\delta_0/w(t)$. We assume diffusive broadening, $w^2 - \delta_0^2 \propto t$. Matching its initial correction to Equation 86 gives

$$w^2(t) = \delta_0^2 + 2t \left(\overbrace{\kappa_\nu \nu}^{\text{viscosity}} + \underbrace{\kappa_\eta \eta}_{\text{resistivity}} \right). \tag{89}$$

Then

$$\frac{\delta_0}{w(t)} = \left[1 + \frac{2t(\kappa_\nu \nu + \kappa_\eta \eta)}{\delta_0^2}\right]^{-1/2} = 1 - \frac{t(\kappa_\nu \nu + \kappa_\eta \eta)}{\delta_0^2} + \cdots. \tag{90}$$

The fixed-shape profile and diffusive width give the inverse square root. Matching the initial slope fixes the factor of two. The evolving shear, field, and density need not share this shape or width.

2.6.3 Resistive damping in a broadening layer

For fixed velocity jump, a representative shear scales as $\Delta U/w(t)$. Its square is reduced by $\delta_0^2/w^2(t)$, motivating the replacement

$$\exp\left[-\frac{\beta \eta t^3}{\delta_0^2 t_\Delta^2}\right] \quad \longrightarrow \quad \exp\left[-\frac{\beta \eta t^3}{t_\Delta^2 w^2(t)}\right]. \tag{91}$$

This preserves the leading cubic loss and reduces subsequent losses as the layer broadens. For an evolving shear, $k_x = p_x - p_y \int_0^t \partial_x U(t') dt'$, and the exact local damping depends on $\int_0^t k^2(t') dt'$. Using the width at time t approximates this history. The finite-time form is therefore approximate, with all coefficients fixed by the initial conditions.

2.6.4 Energy evolution at finite Reynolds numbers

Diffusion of the initial field damps all three ideal contributions. Layer broadening and the additional cubic loss modify only the quadratic winding term. Combining these factors gives

$$\frac{E_{\mathrm{mag}}(t)}{E_{\mathrm{mag},0}} \simeq \overbrace{e^{-\alpha t/(\mathrm{Rm}_\Delta t_\Delta)}}^{\text{diffusion of seed field}} \Bigg[1 + a\frac{t}{t_\Delta} + b(t/t_\Delta)^2 \underbrace{\left[1 + 2(t/t_\Delta)\left(\frac{\kappa_\nu}{\mathrm{Re}_\Delta} + \frac{\kappa_\eta}{\mathrm{Rm}_\Delta}\right)\right]^{-1/2}}_{\text{viscous and resistive broadening}} \times \underbrace{\exp\left(-\frac{\beta(t/t_\Delta)^3}{\mathrm{Rm}_\Delta[1 + 2(t/t_\Delta)(\kappa_\nu/\mathrm{Re}_\Delta + \kappa_\eta/\mathrm{Rm}_\Delta)]}\right)}_{\text{resistive loss from winding-created gradients}}\Bigg]. \tag{92}$$

Here $\nu = \delta_0^2/(\mathrm{Re}_\Delta t_\Delta)$ and $\eta = \delta_0^2/(\mathrm{Rm}_\Delta t_\Delta)$ have been substituted from Equation 2. The linear term includes only diffusion of the initial field. As $\mathrm{Re}_\Delta, \mathrm{Rm}_\Delta \to \infty$, the model recovers the ideal winding polynomial in Equation 19.

2.7 Winding growth rates

Winding produces algebraic energy growth. To compare the model with the simulations, we distinguish its instantaneous logarithmic growth rate from a slope fitted over a finite interval. The instantaneous rate is

$$\gamma_{\mathrm{wind}}(t) = \frac{d\ln E_{\mathrm{mag}}(t)}{dt}. \tag{93}$$

For ideal winding,

$$\gamma_{\mathrm{wind}}(t) = \frac{1}{t_\Delta}\frac{a + 2b(t/t_\Delta)}{1 + a(t/t_\Delta) + b(t/t_\Delta)^2}. \tag{94}$$

When the quadratic term dominates, $\gamma_{\mathrm{wind}} \simeq 2/t$. A fitted slope over a short interval summarizes this changing rate. A free-intercept fit minimizes the time integral of $[\ln(E_{\mathrm{mag}}/E_{\mathrm{mag},0}) - c - \gamma_{\mathrm{wind,fit}} t]^2$ on $[t_1, t_2]$. Its equally weighted slope is

$$\gamma_{\mathrm{wind,fit}} = \frac{\int_{t_1}^{t_2} (t - \bar{t}) \ln[E_{\mathrm{mag}}(t)/E_{\mathrm{mag},0}]\, dt}{\int_{t_1}^{t_2} (t - \bar{t})^2 dt}, \qquad \bar{t} = \frac{t_1 + t_2}{2}. \tag{95}$$

Model and simulation are fitted over the same interval $[t_1, t_2]$, with equal weighting in time, and report $\gamma_{\mathrm{wind,fit}} t_\Delta$. This finite-interval slope does not imply exponential growth. A field-amplitude rate inferred from $\sqrt{E_{\mathrm{mag}}}$ is half the energy rate.

3 Ideal spectral model

The ideal mode solution also determines how the generated energy is distributed in wavenumber. The spectral density follows from the energy assigned to each group of sheared modes.

3.1 Assumptions for the ideal spectrum

The ideal spectrum follows from the magnetic mode solution for a stationary, locally linear shear, with $\eta = 0$. We assume strong winding, $|\partial_x U|t \gg 1$, at $k \gg k_0$. The seed is statistically homogeneous and isotropic, with energy initially concentrated at $|\boldsymbol{p}| = k_0$. The two field directions perpendicular to each wavevector carry equal mean energy.

A smooth spectrum averages over continuously distributed seed directions. These assumptions fix its explicit normalization, whereas a, b retain the actual finite seed. We now use the mode mapping to convert the initial directional energy into energy per unit k.

3.2 Wavenumber and energy transport

The mode solution specifies both the destination wavenumber and the generated energy. In the strongly wound range, Equation 27 gives

$$k \simeq |k_x| \simeq |\partial_x U|t|p_y|. \tag{96}$$

This requires $|\partial_x U|t|p_y| \gg k_0$, so the generated k_x exceeds all initial components. Outside this range, the initial wavevector components must be retained.

At the same time, Equation 14 gives

$$|\widehat{B}_y|^2 \simeq (\partial_x U)^2 t^2 |\widehat{B}_{x0}|^2. \tag{97}$$

Thus a magnetic mode's generated energy grows as t^2, while its wavenumber grows as t. We group these energies by their initial along-flow wavenumber: $F(q)dq$ denotes the initial x-component magnetic energy in magnetic modes with $q < |p_y| < q + dq$, where $q = |p_y| \geq 0$. Their generated energy is

$$E_{\rm mag}(k, t)\, dk \simeq (\partial_x U)^2 t^2 F(q)\, dq. \tag{98}$$

Since $k = |\partial_x U|tq$ at fixed t, $dk = |\partial_x U|t\, dq$. Dividing by this widening wavenumber interval yields

$$\boxed{E_{\rm mag}(k, t) \simeq |\partial_x U|t\, F\left(\frac{k}{|\partial_x U|t}\right).} \tag{99}$$

The spectral density gains one power of t because the wavenumber interval containing a fixed set of magnetic modes also grows as t.

3.3 Initial energy distribution across magnetic modes

The distribution $F(q)$ fixes the spectrum's shape and normalization. Isotropic mode directions and the divergence-free constraint determine this distribution for the assumed seed.

3.3.1 Distribution of along-flow wavenumbers

Uniform mode directions determine the energy distribution over $q = |p_y|$. For a vector uniformly distributed over a sphere, the area between polar angles θ and $\theta + d\theta$ is proportional to $\sin\theta\, d\theta$. With y as the polar axis, $p_y = k_0 \cos\theta$, so p_y is uniformly distributed between $-k_0$ and k_0. Folding

the positive and negative parts together makes $q = |p_y|$ uniform on $[0, k_0]$. The *total* initial magnetic energy in dq is therefore $E_{\rm mag,0}\, dq/k_0$.

Only the initial x component generates the dominant B_y. For $\mu = q/k_0$, the unit wavevector has

$$\widehat{p}_x = \sqrt{1-\mu^2}\cos\phi, \tag{100}$$

where ϕ is the angle around the flow direction, y.

3.3.2 Magnetic-component fraction

To obtain $F(q)$, the total energy at each q must be weighted by its x-component fraction. The constraint $\boldsymbol{p}\cdot\widehat{\boldsymbol{B}}_0 = 0$ requires the field to lie perpendicular to $\boldsymbol{p}$. For equal power in the two directions in that plane, the fraction of magnetic energy in x is

$$\frac{1-\widehat{p}_x^2}{2}. \tag{101}$$

The numerator is the squared length of the projection of $\boldsymbol{e}_x$ onto that plane. The denominator counts its two equally weighted directions. Averaging over ϕ, using $\langle\cos^2\phi\rangle_\phi = 1/2$, gives

$$\frac{1}{2}\left[1-\frac{1-\mu^2}{2}\right] = \frac{1+\mu^2}{4}. \tag{102}$$

Multiplying this fraction by the total energy distribution gives

$$\boxed{F(q) = \frac{E_{\rm mag,0}}{4k_0}\left[1+\left(\frac{q}{k_0}\right)^2\right], \qquad 0<q<k_0.} \tag{103}$$

As a normalization check,

$$\int_0^{k_0} F(q)dq = \frac{E_{\rm mag,0}}{4}\left(1+\frac{1}{3}\right) = \frac{E_{\rm mag,0}}{3}, \tag{104}$$

the expected initial energy in one component of an isotropic field.

3.4 Winding spectrum

The derived $F(q)$ now determines the winding spectrum. Substitution into Equation 99 gives

$$E_{\rm mag}(k,t) \simeq \frac{E_{\rm mag,0}}{4k_0}|\partial_x U|t\left[1+\left(\frac{k}{|\partial_x U|tk_0}\right)^2\right]. \tag{105}$$

For strongly wound modes well below the expanding front,

$$k_0 \ll k \ll |\partial_x U|tk_0, \tag{106}$$

the second term in brackets is small. We obtain

$$\boxed{E_{\rm mag}(k,t) \simeq \frac{E_{\rm mag,0}}{4k_0}|\partial_x U|t \propto k^0.} \tag{107}$$

"Flat" refers to energy per unit k, as defined in Equation 4. Energy per unit $\ln k$ is $kE_{\rm mag}(k,t)$ and would rise linearly with k in the same band. The bracket in Equation 105 rises from one in the interior to two at the expanding front. Near the seed wavelengths and the exact front, the neglected wavevector components also matter.

3.4.1 Total generated magnetic energy

Integrating the spectrum must recover the leading generated energy. Using Equation 99 and changing variable back to q gives

$$\int_0^{|\partial_x U| t k_0} E_{\rm mag}(k,t)dk \simeq (\partial_x U)^2 t^2 \int_0^{k_0} F(q)dq \tag{108}$$

$$= \frac{E_{\rm mag,0}}{3}(\partial_x U)^2 t^2. \tag{109}$$

The integral recovers the leading quadratic energy for a constant shear and isotropic seed. Extending the approximate spectrum to $k = 0$ checks this leading normalization, but does not resolve the remaining low-k initial field. Extending the flat interior alone to the edge would give $E_{\rm mag,0}(\partial_x U)^2 t^2/4$. The bracket in Equation 105 supplies the rest of the leading energy.

3.4.2 Seeds with multiple wavelengths

The same calculation extends to seeds with power over several initial wavelengths. Here $E_{\rm mag}(p,0)dp$ denotes the energy in $[p, p+dp]$. Adding the single-radius result for all $p \geq q$ gives

$$F(q) = \int_q^\infty \frac{E_{\rm mag}(p,0)}{4p}\left[1 + \left(\frac{q}{p}\right)^2\right] dp. \tag{110}$$

If the seed is concentrated at finite, nonzero p, then

$$F(0) = \frac{1}{4}\int_0^\infty \frac{E_{\rm mag}(p,0)}{p}\, dp \tag{111}$$

is finite and nonzero, giving a flat interior of height $|\partial_x U| t F(0)$. A finite anisotropic seed instead supplies a discrete F from its actual component energies and $|p_y|$ values.

3.5 Evolution of energy per unit wavenumber

The mode mapping also gives an energy balance in wavenumber. A strongly wound magnetic mode obeys $dk/dt = k/t$ and gains energy at fractional rate $2/t$. For a fixed interval $[k_1, k_2]$, transport across its boundaries and stretching give

$$\frac{d}{dt}\int_{k_1}^{k_2} E_{\rm mag}(k,t)\, dk = -\left[\frac{k}{t}E_{\rm mag}(k,t)\right]_{k_1}^{k_2} + \int_{k_1}^{k_2}\frac{2}{t}E_{\rm mag}(k,t)\, dk. \tag{112}$$

Because k_1, k_2 are arbitrary, the integrands satisfy

$$\partial_t E_{\rm mag}(k,t) = -\partial_k\left[\frac{k}{t}E_{\rm mag}(k,t)\right] + \frac{2}{t}E_{\rm mag}(k,t). \tag{113}$$

Expanding the derivative makes the distinction between total magnetic mode energy and energy per unit k explicit:

$$\partial_t E_{\rm mag} + \frac{k}{t}\partial_k E_{\rm mag} = \frac{1}{t}E_{\rm mag}. \tag{114}$$

Along $dk/dt = k/t$, this gives $d\ln E_{\rm mag}/dt = 1/t$ and hence Equation 99. The equation applies only after $|\partial_x U| t \gg 1$, which excludes its apparent singularity at $t = 0$. Each initial magnetic mode follows a unique $k(t)$. The evolution equation therefore contains transport and stretching without random spreading in wavenumber.

3.5.1 Resistive cutoff

Restoring resistivity limits the smallest surviving wavelength. Its effect on the flat band follows from the cubic term in Equation 33. With $k \simeq |\partial_x U| t q$,

$$\frac{2}{3}\eta(\partial_x U)^2 q^2 t^3 \simeq \frac{2}{3}\eta k^2 t. \tag{115}$$

Thus the strong-winding resistive spectrum is approximately

$$E_{\mathrm{mag}}(k,t) \simeq |\partial_x U| t F\left(\frac{k}{|\partial_x U| t}\right) e^{-2\eta k^2 t/3}. \tag{116}$$

The same result follows by adding $-2\eta k^2 E_{\mathrm{mag}}$ to Equation 113. Damping is weak when $2\eta k^2 t/3 \ll 1$, so the flat band requires

$$k_0 \ll k \ll \min\left[|\partial_x U| t k_0, \left(\frac{3}{2\eta t}\right)^{1/2}\right]. \tag{117}$$

The upper limits are the expanding front of wound modes and the resistive cutoff, $k_\eta = (3/2\eta t)^{1/2}$, respectively. The cutoff recedes to infinity as $\eta \to 0$.